

THE EXPONENTIAL MAP FOR THE UNITARY GROUP $SU(2,2)$

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Abstract : In this article we extend our previous results for the orthogonal group, $SO(2,4)$, to its homomorphic group $SU(2,2)$. Here we present a closed, finite formula for the exponential of a 4×4 traceless matrix, which can be viewed as the generator (Lie algebra elements) of the $SL(4, C)$ group. We apply this result to the $SU(2,2)$ group, which Lie algebra can be represented by the Dirac matrices, and discuss how the exponential map for $SU(2,2)$ can be written by means of the Dirac matrices.

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1. INTRODUCTION

In a previous paper by these authors [Barut, Zeni and Laufer], we obtained a closed, finite formula for the exponential map from the Lie algebra into the defining representation of the orthogonal groups, and in particular for the $SO_+(2, 4)$ group. This result is a generalization of the well known and important formulas for the $SO(3)$ group [Barut, 80], and of the analogous result for the Lorentz group, $SO_+(1, 3)$ [Zeni and Rodrigues, 90].

The present article deals with the exponential map from the Lie algebra into the defining representation of $SU(2, 2)$ group, which is the covering group of the $SO_+(2, 4)$ group. The result presented here can be viewed as a generalization of the recent result to the $SL(2, C)$ group [Zeni and Rodrigues, 92],

$$e^F = \cosh z + \hat{F} \sinh z \quad (1.1)$$

where $F = (e_i + \mathbf{i} b_i)\sigma_i$ is a complex vector expanded in the Pauli matrices, the complex variable z is such that $z^2 = F^2$, and $\hat{F} = F/z$. We remark that the above result contains the particular case of the $SU(2)$ group, when $e_i = 0$.

The group $SU(2, 2)$ and its homomorphic $SO_+(2, 4)$ has several applications in theoretical physics [Barut and Brittin]. For instance, it is the largest group that leaves the Maxwell equations invariant [Bateman]; from its subgroups, $SO(3)$ and $SO(2, 1)$, we can obtain the whole spectrum of the hydrogen atom as predicted by Schrödinger theory [Barut, 72]; and more recently it has been used in spin gauge theories as an attempt to generalize the minimal coupling [Barut and McEwan], [Chisholm and Farwell], [Dehnen and Ghaboussi]. Also, the unitary groups play an important role in Quantum Mechanics [Barut and Raczka], and we can find in the literature several articles dealing with the parametrizations of these groups (see [Barnes and Delbourgo] and references therein). For instance, [Bincer] presents a parametrization for the exponential through a set of orthonormal vectors that must be computed from the diagonal form of the matrix. We remark that, if one uses our method to obtain the exponential [Barut, Zeni and Laufer], we need only to compute the eigenvalues of the matrix, no further computations are involved, such as the eigenvectors.

We remark that, besides its application in group theory, the exponential of a matrix has an important application in the solution of a system of differential equations, as discussed in [Barut, Zeni and Laufer], so the method developed in this present paper and in our previous paper can be useful for that study. A comprehensive review of methods to exponentiate an arbitrary matrix is given in [Moler and van Loan] and references therein.

The plan of the present paper is the following: in Section II we obtain the exponential of a 4×4 traceless matrix; in Section III we discuss some particular cases of that exponential;

in Section IV we study the representation of the exponential in the Dirac algebra, in particular, the cases when the generator is either the sum of a vector and a axial vector, or a bivector; in Section V we make our final comments.

2. THE EXPONENTIAL OF A 4×4 TRACELESS MATRIX

The method presented previously by these authors [Barut, Zeni and Laufer] can be generalized straightforward to exponentiate any given matrix. Basically, the algorithm presented was based on the Hamilton-Cayley theorem and it emphasizes that the exponential must be written by means of the eigenvalues. Also remarkable is the use of the (square-root) discriminant related to the characteristic equation of the matrix, as a multiplier to simplify the expressions of the coefficients that appears in the recurrence relation. Finally, we analyse the coefficients looking at each eigenvalue separately.

We are going to apply the above algorithm to the generators of $SL(4, C)$. We recall that the Lie algebra of $SL(n, C)$ is defined by,

$$sl(n, C) = \{ H \in C(n), \text{ such that, } e^H \in SL(n, C) \} \quad (2.1)$$

where $C(n)$ is the space of $n \times n$ complex matrices.

Therefore the generators of $SL(n, C)$ are traceless, since we have $\det e^H = e^{\text{Tr } H}$.

From now on we are restricting ourselves to 4×4 matrices.

The characteristic equation of a 4×4 matrix is given by,

$$\begin{aligned} \det(H - \lambda I) &= \lambda^4 - d_0 \lambda^3 - a_0 \lambda^2 - b_0 \lambda - c_0 \\ &= (\lambda - w)(\lambda - x)(\lambda - y)(\lambda - z) = 0 \end{aligned} \quad (2.2)$$

where w, x, y and z are the eigenvalues.

The matrices representing the generators of $SL(4, C)$ are traceless, so in this case we have that the sum of the eigenvalues vanish,

$$d_0 = w + x + y + z = \text{Tr } H = 0 \quad (2.3)$$

The recurrence relations

The Cayley-Hamilton theorem says that a matrix satisfies a matrix equation identical to its characteristic equation, therefore we can write all higher powers of H in terms of the first powers [Barut, Zeni and Laufer]

$$H^{(4+i)} = d_i H^3 + a_i H^2 + b_i H + c_i \quad (2.4)$$

So the series for e^H becomes series in the coefficients of the above equation. Multiplying the recurrence relation, eq.(2.4), by H , and using the Hamilton-Cayley theorem related to eq.(2.2), one obtains recurrence relations for the coefficients, which holds for $i \geq 0$,

$$a_{i+1} = b_i + d_i a_0, \quad b_{i+1} = c_i + d_i b_0, \quad c_{i+1} = d_i c_0, \quad d_{i+1} = a_i, \quad (2.5)$$

From the above relations we can also show that $(i \geq 2)$,

$$a_{i+2} = a_i a_0 + a_{i-1} b_0 + a_{i-2} c_0 \quad (2.6)$$

In the next step, as outlined in [Barut, Zeni and Laufer], we introduce the *square-root of the discriminant* of eq.(2.2), indicated hereafter by m ,

$$m = (w - x)(w - y)(w - z)(x - y)(x - z)(y - z) \quad (2.7)$$

We write the first coefficients, a_0 , b_0 and c_0 by means of the eigenvalues, according to eq.(2.2), and from eq.(2.6) we find that the general term for the coefficients a_i , multiplied by m , is given by $(i \geq 0)$

$$m a_i = t(w, y, z) x^{5+i} + t(x, w, z) y^{5+i} + t(w, x, y) z^{5+i} + t(y, x, z) w^{5+i} \quad (2.8)$$

where we made use of the *alternating*, $t(w, y, z)$, function of three variables,

$$t(w, y, z) = (w - y)(y - z)(z - w) \quad (2.9)$$

Now, based on eq.(2.8), the series for the coefficients can be summed up easily, since we can write the other coefficients by means of a_i , according to eq.(2.5). For instance,

$$b_{i+1} = a_{i-2} c_0 + a_{i-1} b_0 \quad (2.10)$$

In order to write down the exponential, it is convenient to introduce the *symmetric*, $s(w, y, z)$, function in three variables and the *product* of the symmetric and alternating functions, indicated hereafter by $st(w, y, z)$,

$$s(w, y, z) = w y + w z + y z, \quad st(w, y, z) = s(w, y, z) t(w, y, z) \quad (2.11)$$

The closed, finite formula for the exponential of a 4×4 traceless matrix

$$\begin{aligned}
m e^H &= -w x y z \left(t(w, y, z) \frac{e^x}{x} + t(x, w, z) \frac{e^y}{y} + t(w, x, y) \frac{e^z}{z} + t(y, x, z) \frac{e^w}{w} \right) 1 \\
&+ \left(st(w, y, z) e^x + st(x, w, z) e^y + st(w, x, y) e^z + st(y, x, z) e^w \right) H \\
&+ \left(x t(w, y, z) e^x + y t(x, w, z) e^y + z t(w, x, y) e^z + w t(y, x, z) e^w \right) H^2 \\
&+ \left(t(w, y, z) e^x + t(x, w, z) e^y + t(w, x, y) e^z + t(y, x, z) e^w \right) H^3
\end{aligned} \tag{2.12}$$

3. SOME SPECIAL CASES OF THE EXPONENTIAL

The case when $b_0 = 0$.

Now we are going to see that the above formula for the exponential simplifies considerably in the case when the characteristic equation for H , eq.(2.2), has no term in the first power, i.e. $b_0 = 0$. In this case the recurrence relations, eq.(2.4), for the even (odd) powers involves only the even (odd) powers.

If $b_0 = 0$ we have a quadratic equation in the square of the eigenvalue of H , so we can set $w = -x$ and $z = -y$, and work only with two eigenvalues, x and y . The Hamilton-Cayley theorem, related to eq.(2.2), becomes,

$$H^4 - (x^2 + y^2) H^2 + x^2 y^2 = 0 \tag{3.1}$$

The square-root of the discriminant, m , eq.(2.7), reduces in this case to,

$$m = -4 x y (x^2 - y^2)^2 \tag{3.2}$$

Therefore, the series for e^H , eq.(2.12), in the case when $b_0 = 0$, is given by,

$$\begin{aligned}
e^H &= \frac{x^2 \cosh y - y^2 \cosh x}{x^2 - y^2} 1 + \frac{\cosh x - \cosh y}{x^2 - y^2} H^2 \\
&+ \frac{x^3 \sinh y - y^3 \sinh x}{x y (x^2 - y^2)} H + \frac{y \sinh x - x \sinh y}{x y (x^2 - y^2)} H^3
\end{aligned} \tag{3.3}$$

Further Simplifications: the case when $H^2 = x^2 1$ — In this case the square of the generator can be identified with the square of one eigenvalue, say x , which is a particular situation of the Hamilton-Cayley equation given above, eq.(3.1). Examples includes the important cases when the generator is either a vector (or axial vector), or a bivector from Dirac algebra. This last case is just the Lorentz group.

Therefore, substituting for $H^2 = x^2$ in eq.(3.3), we obtain,

$$e^H = \cosh x + \frac{\sinh x}{x} H \quad (3.4)$$

This is the formula given in [Zeni and Rodrigues, 92], for the exponential of the generators of the Lorentz group. We remark that [Zeni and Rodrigues, 92] proved in a very simple way that every proper and orthochronous Lorentz transformation can be written as the exponential of some generator.

4. THE GROUP $SU(2,2)$ AND THE DIRAC ALGEBRA

We recall that the Lie algebra of $SU(2,2)$ group is defined by [Kihlberg et al],

$$su(2,2) = \{H \in C(4), \text{ such that, } H^\dagger \beta = -\beta H\} \quad (4.1)$$

with $\beta = \text{diag}(1, 1, -1, -1)$.

Also, the matrix algebra $C(4)$ is isomorphic to the Dirac algebra, therefore the generators of $SU(2,2)$ can be represented by an appropriate set of Dirac matrices.

The *standard* representation for the Dirac matrices is given by,

$$\gamma_0 = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \quad (4.2)$$

where $i \in [1, 3]$ and we set $\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$, so its square is *negative*, i.e., $\gamma_5^2 = -1$, and it is *skew-hermitian*.

A general element of the Lie algebra of $SU(2,2)$ can be written as follows

$$H = iV + F + \gamma_5 A + i c \gamma_5 = i v^\mu \gamma_\mu + F^{\mu\nu} \gamma_{\mu\nu} + \gamma_5 a^\mu \gamma_\mu + i c \gamma_5 \quad (4.3)$$

where $\gamma_{\mu\nu} = \gamma_\mu \gamma_\nu$, $\mu \leq \nu$, $\mu, \nu \in [0, 3]$. The components v^μ , $F^{\mu\nu}$, a^μ , and c are *real* numbers.

Vector ($H = iV$) or Axial Vector ($H = \gamma_5 A$): In this case, the square of the generator is a *real* number,

$$\begin{aligned} \text{if, } H = iV = i v^\mu \gamma_\mu, & \quad \Rightarrow \quad H^2 = -V^2 = -v^\mu v_\mu = x^2 \\ \text{if, } H = \gamma_5 A = \gamma_5 a^\mu \gamma_\mu, & \quad \Rightarrow \quad H^2 = A^2 = a^\mu a_\mu = x^2 \end{aligned} \quad (4.4)$$

In this case, the eigenvalues presents in eq.(3.1) are equal to each other, i.e., we have $x^2 = y^2$.

The previous formula for the exponential of H , eq.(3.4), holds in every case, i.e., V can be lighth-like ($V^2 = 0$), space-like ($V^2 < 0$), or time-like ($V^2 > 0$).

Bivector ($H = F$): The square of a bivector in the Dirac algebra can be *formally* identified with a complex number, which is one of the eigenvalues, we say x . The other eigenvalue y is related to x by complex conjugation. The imaginary unit is represented by γ_5 .

Let us write the bivector as follows,

$$F = F^{\mu\nu} \gamma_{\mu\nu} = (e^i + \gamma_5 b^i) \gamma_{i0} \quad (4.5)$$

In the latter form it is easy to compute the square of F [Zeni and Rodrigues, 92],

$$F^2 = \vec{e}^2 - \vec{b}^2 + 2 \gamma_5 \vec{e} \cdot \vec{b} \quad (4.6)$$

From the above equation we can deduce the explicit form of the Hamilton-Cayley theorem in this case,

$$F^4 - 2(\vec{e}^2 - \vec{b}^2) F^2 + 4(\vec{e} \cdot \vec{b})^2 + (\vec{e}^2 - \vec{b}^2)^2 = 0 \quad (4.7)$$

If we compare eq.(4.7) with eq.(3.1), we find that the eigenvalues are given by,

$$x^2 = \vec{e}^2 - \vec{b}^2 + 2 \mathbf{i} \vec{e} \cdot \vec{b}, \quad y^2 = \vec{e}^2 - \vec{b}^2 - 2 \mathbf{i} \vec{e} \cdot \vec{b} \quad (4.8)$$

Observe that the expression for x is the same expressions as that for F^2 . We have only to replace the imaginary unit for γ_5 . Therefore, eq.(3.4) applies for the exponential of a bivector.

The Sum of a Vector and an Axial Vector ($H = W = iV + \gamma_5 A$): In this case we are going to show that the fourth power of the generator can be written by means of the second power and the identity, so eq.(3.3) applies for the exponential of a V - A generator. Also, we obtain an explicit expression for the second and third power by means of V and A , eq.(4.16), which can simplify further computations with the exponential, eq.(3.3). From now on, we are indicating $W = H$ for the generator, so we have,

$$W = iV + \gamma_5 A = (iV^\mu + \gamma_5 A^\mu) \gamma_\mu \quad (4.9)$$

Computing the second and fourth power of W we find,

$$W^2 = A^2 - V^2 + i \gamma_5 (A \cdot V - V \cdot A) \quad (4.10)$$

$$W^4 = 2(A^2 - V^2) W^2 + 4(A \bullet V)^2 - (A^2 + V^2)^2 \quad (4.11)$$

Therefore, a V-A generator satisfies eq.(3.1) and it can be exponentiated as in eq.(3.3). Observe that if, $V = A$, i.e., $W = (1 + \gamma_5)V$, it implies $W^2 = 0$, and $e^W = 1 + W$. We introduced $A \bullet V$ as the *inner product* in the Dirac algebra (also $V^2 = V \bullet V$),

$$V \bullet A = 1/2 (V A + A V) = v^\mu a_\mu$$

We also remark that $(A V - V A)^2 = 4 (A \bullet V)^2 - 4 A^2 V^2 \stackrel{\text{def}}{=} \Delta$.

If we compare eq.(4.11) with eq.(3.1) we see that the eigenvalues are given by,

$$x^2 = A^2 - V^2 + \sqrt{\Delta}, \quad y^2 = A^2 - V^2 - \sqrt{\Delta} \quad (4.12)$$

To get a convenient expression for the third power of W we introduce a new element, W^* , defined by,

$$W^* = A + i \gamma_5 V \quad (4.13)$$

The products $W W^*$ and $W^* W$ will be called here *bicomplex numbers*, i.e., we have now two imaginary *commutative* units, γ_5 and i . Moreover, the above products are related to each other through complex conjugation respective to γ_5 , i.e., we have,

$$\begin{aligned} u &\stackrel{\text{def}}{=} W W^* = 2 i A \bullet V + (A^2 + V^2) \gamma_5 \\ \bar{u} &\stackrel{\text{def}}{=} W^* W = 2 i A \bullet V - (A^2 + V^2) \gamma_5 \end{aligned} \quad (4.14)$$

It is remarkable that the product $u \bar{u} = \bar{u} u$ is a real number, which is just the *determinant* of $H = W$ in the matrix representation (cf. eq.(4.11) above),

$$u \bar{u} = \bar{u} u = -4 (A \bullet V)^2 + (A^2 + V^2)^2 \quad (4.15)$$

Based on eq.(4.15), we obtain the inverse of W as (when there is an inverse, i.e., $u \bar{u} \neq 0$),

$$W^{-1} = W^* u^{-1} = \frac{W^* \bar{u}}{u \bar{u}} \quad (4.16)$$

To verify that the above expression just defines the *bilateral* inverse, we call attention to the fact that γ_5 anticommutes with W , so we have $\bar{u} W = W u$. Now considering that $W^3 = W^4 W^{-1}$, it follows from eq.(4.11) and eq.(4.16) that the third power of W is given by (if, $\bar{u} u \neq 0$),

$$W^3 = 2(A^2 - V^2) W - W^* \bar{u} \quad (4.17)$$

which is easily expressed by means of the Dirac matrices, through eq.(4.9) and eq.(4.13).

5. CONCLUSIONS

In this article we presented a finite, closed formula for the exponential of a 4×4 traceless matrix, eq.(2.12). It can be viewed as the exponential of a generator of the $SL(4, C)$ group, which includes the $SU(2, 2)$ group as a subgroup. Our approach to get the exponential is based on our previous work [Barut, Zeni and Laufer].

Eq.(2.12) is a generalization of the exponential for generators of the $SL(2, C)$ group presented in [Zeni and Rodrigues, 92].

The finite formula for the exponential, eq.(2.12), involves only the computations of the eigenvalues and the first three powers of the matrix, no further computations (e.g., eigenvectors) are needed to obtain the exponential (cf. [Moler and van Loan]).

We have also presented some special cases of this exponential, eq.(3.3) and eq.(3.4). They include the important cases when the generator of the $SU(2, 2)$ group is identified either with a bivector or the sum of a vector and a axial vector, as discussed in Section IV. For both cases, we gave explicit expressions for the eigenvalues of the generators, eq.(4.8) and eq.(4.12), as derived from Dirac algebra. Moreover, in the case of a V-A generator, we obtained a simple expression for the third power of the generator by means of V and A , eq.(4.17), which is needed to exponentiate the generator (see eq.(3.3)).

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